Lecture 25

Dynamic Programming: LCS (contd.)

Source: Introduction to Algorithms, CLRS

Optimal Substructure in LCS

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LCS of $x_1 x_2 \ldots x_{m-1}$ and $y_1 y_2 \ldots y_{n-1}$ followed by x_m

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Since $x_m \neq y_n$, W cannot end with both x_m and y_n.

If W doesn't end with x_m (WLOG), then W will be an **CS** of $x_1x_2\ldots x_{m-1}$ and $y_1y_2\ldots y_n$.

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This is a contradiction as W is longer than $\mathbf{LCS}(x_1x_2\ldots x_{m-1}, y_1y_2\ldots y_n)$.

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Claim: Let $X = x_1x_2\ldots x_m$ and $Y = y_1y_2\ldots y_n$ be sequences such that $x_m \neq y_n$. Then, at least one out of $\mathbf{LCS}(x_1x_2\ldots x_{m-1}, y_1y_2\ldots y_n)$ and $\mathbf{LCS}(x_1x_2\ldots x_m, y_1y_2\ldots y_{n-1})$ will be an $\mathbf{LCS}(X, Y)$.

Proof: Suppose neither $\mathbf{LCS}(x_1x_2\ldots x_{m-1}, y_1y_2\ldots y_n)$ nor $\mathbf{LCS}(x_1x_2\ldots x_m, y_1y_2\ldots y_{n-1})$ is an $\mathbf{LCS}(X, Y)$.

Let $\mathbf{LCS}(x_1x_2\ldots x_{m-1}, y_1y_2\ldots y_n)$ be of length j.

Let $\mathbf{LCS}(x_1x_2\ldots x_m, y_1y_2\ldots y_{n-1})$ be of length k.

Let W be an $\mathbf{LCS}(X, Y)$ of length $l > \mathbf{Max}(j, k)$.

Since $x_m \neq y_n$, W cannot end with both x_m and y_n.

$$X = x_1x_2x_3\ldots x_{m-1}\cancel{x_m}$$

$$Y = y_1y_2y_3\ldots y_{n-1}y_n$$

$$W = w_1w_2w_3\ldots w_{l-1}w_l$$

If W doesn't end with x_m (WLOG), then W will be an $\mathbf{CS}$ of $x_1x_2\ldots x_{m-1}$ and $y_1y_2\ldots y_n$.

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length l

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■

Recurrence for LCS

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Let $X = x_1 x_2 \ldots x_m$ and $Y = y_1 y_2 \ldots y_n$ be two sequences.

Recurrence for LCS

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Recurrence for LCS

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Let $LCS(i, j)$ = length of the **LCS** of $x_1 x_2 \ldots x_i$ and $y_1 y_2 \ldots y_j$. Then,

$$LCS(i,j) = \begin{cases} 0, & \text{if } i = 0 \text{ or } j = 0 \\ \\ \\ \end{cases}$$

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Overlapping Subproblems in LCS

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LCS(5,8)

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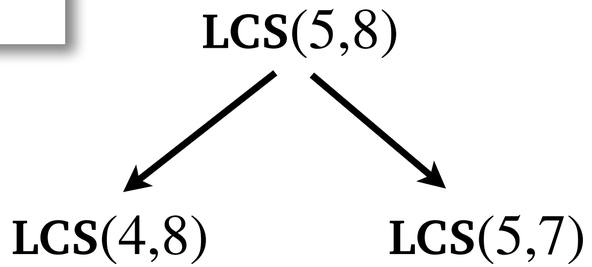

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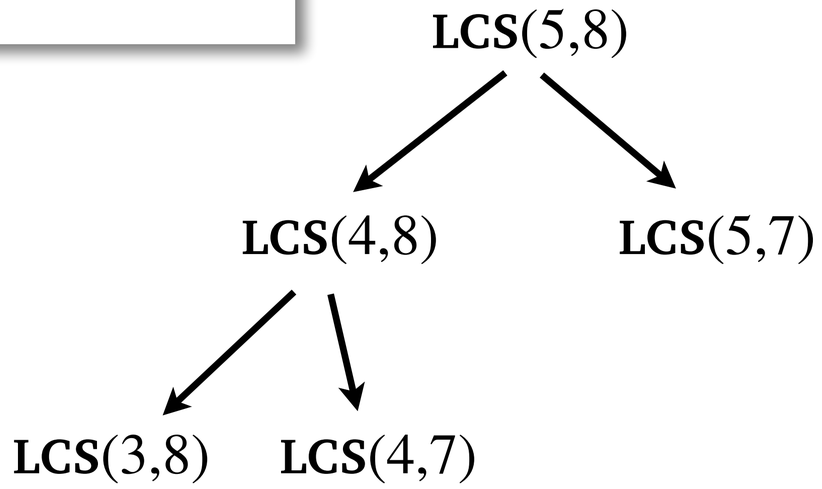

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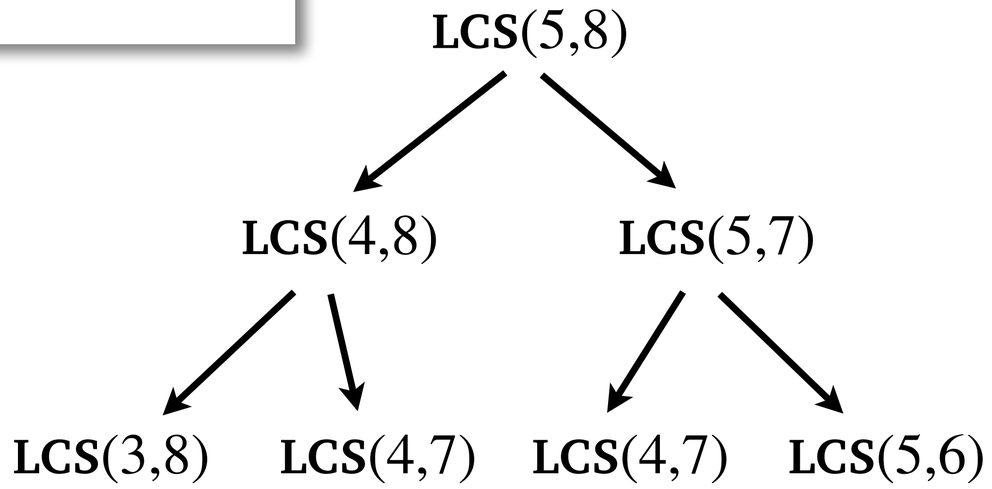

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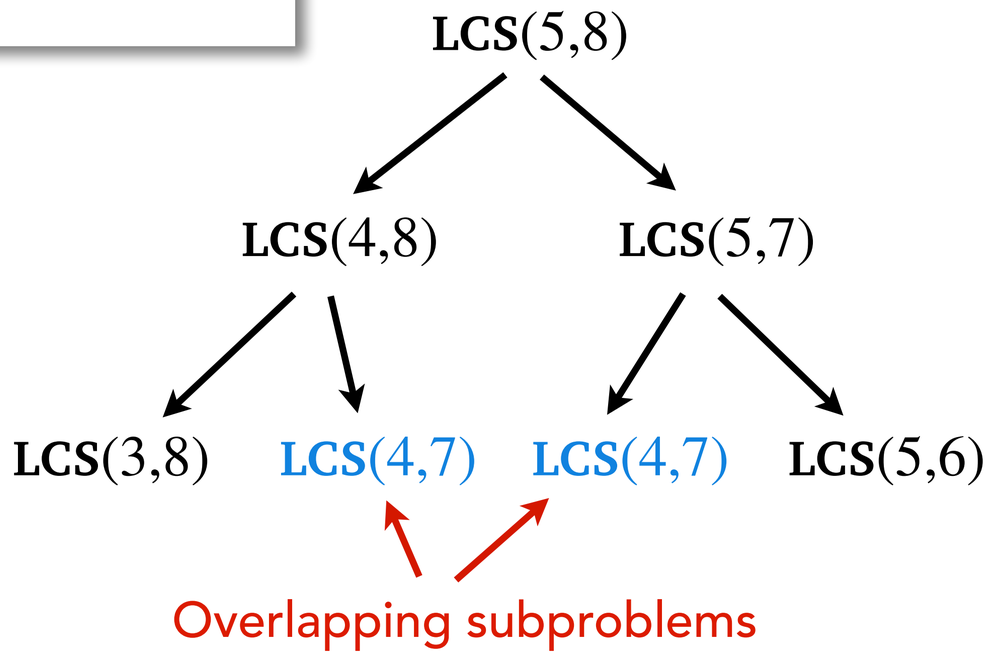

Top-Down DP for LCS

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$dp[0:n][0:m] = \{-1, -1, \ldots, -1\}$

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$dp[i][j]$ stores $LCS(i, j)$

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dp

	0	0	0	0	0	0	0
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n 0	0	-1	-1	-1	-1	-1	-1
0	0	-1	-1	-1	-1	-1	-1
0	0	-1	-1	-1	-1	-1	-1
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Call with **LCS**$(|X|, |Y|)$

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Time Complexity: $O(nm)$

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Why?

Bottom-Up DP for LCS

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LCS(X, Y)

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dp

	0	0	0	0	0	0	0
	0	-1	-1	-1	-1	-1	-1
n	0	-1	-1	-1	-1	-1	-1
	0	-1	-1	-1	-1	-1	-1
	0	-1	-1	-1	-1	-1	-1
	0	-1	-1	-1	-1	-1	-1

m

Bottom-Up DP for LCS

LCS(X, Y)

1. $dp[0:n][0:m] = \{-1, -1, \ldots, -1\}$
2. $dp[0][0:m] = 0, dp[0:n][0] = 0$
3. **for** $i = 1$ **to** n

dp

0	0	0	0	0	0	0
0	-1	-1	-1	-1	-1	-1
0	-1	-1	-1	-1	-1	-1
0	-1	-1	-1	-1	-1	-1
0	-1	-1	-1	-1	-1	-1
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n

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dp

0	0	0	0	0	0	0
0	-1	-1	-1	-1	-1	-1
0	-1	-1	-1	-1	-1	-1
0	-1	-1	-1	-1	-1	-1
0	-1	-1	-1	-1	-1	-1
0	-1	-1	-1	-1	-1	-1

n

m

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4. **for** $j = 1$ **to** m
5. **if** $x_i == y_j$

dp

0	0	0	0	0	0	0
0	-1	-1	-1	-1	-1	-1
0	-1	-1	-1	-1	-1	-1
0	-1	-1	-1	-1	-1	-1
0	-1	-1	-1	-1	-1	-1
0	-1	-1	-1	-1	-1	-1

n

m

Bottom-Up DP for LCS

LCS(X, Y)

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2. $dp[0][0:m] = 0, dp[0:n][0] = 0$
3. **for** $i = 1$ **to** n
4. **for** $j = 1$ **to** m
5. **if** $x_i == y_j$
6. $dp[i][j] = dp[i-1][j-1] + 1$

dp

0	0	0	0	0	0	0
0	-1	-1	-1	-1	-1	-1
0	-1	-1	-1	-1	-1	-1
0	-1	-1	-1	-1	-1	-1
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n

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0	-1	-1	-1	-1	-1	-1
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7. **else**
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dp

0	0	0	0	0	0	0
0	-1	-1	-1	-1	-1	-1
0	-1	-1	-1	-1	-1	-1
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m

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9. **return** $dp[n][m]$

dp

n

0	0	0	0	0	0	0
0	-1	-1	-1	-1	-1	-1
0	-1	-1	-1	-1	-1	-1
0	-1	-1	-1	-1	-1	-1
0	-1	-1	-1	-1	-1	-1
0	-1	-1	-1	-1	-1	-1

m

Bottom-Up DP for LCS

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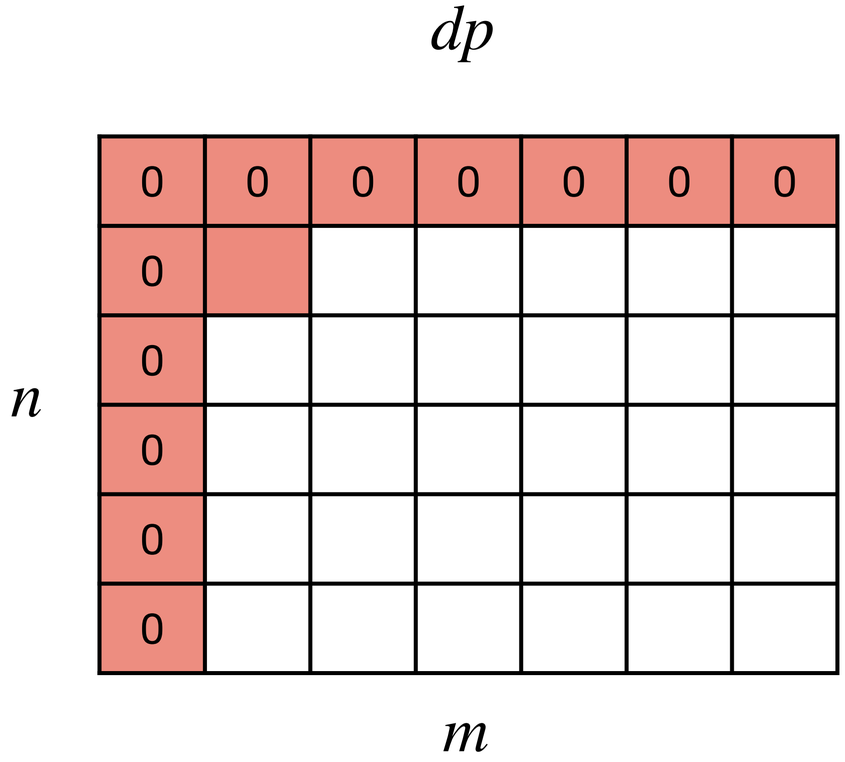

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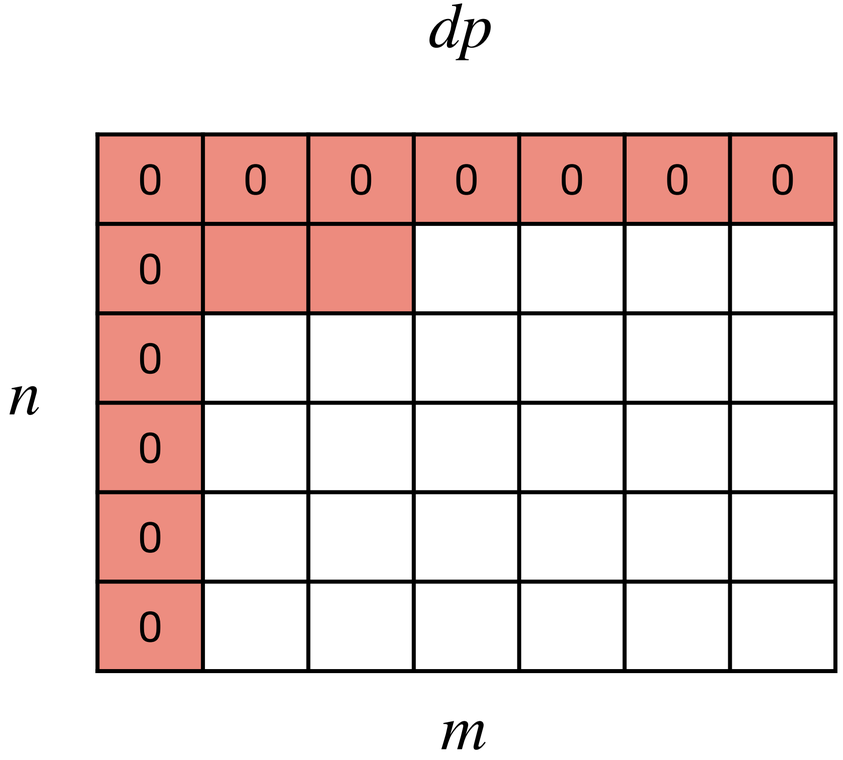

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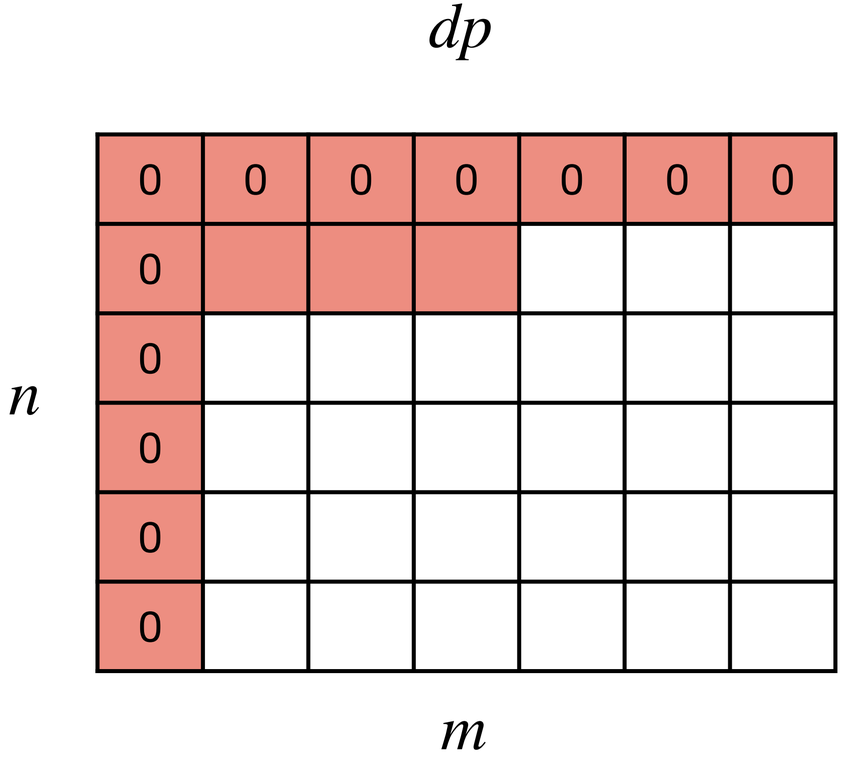

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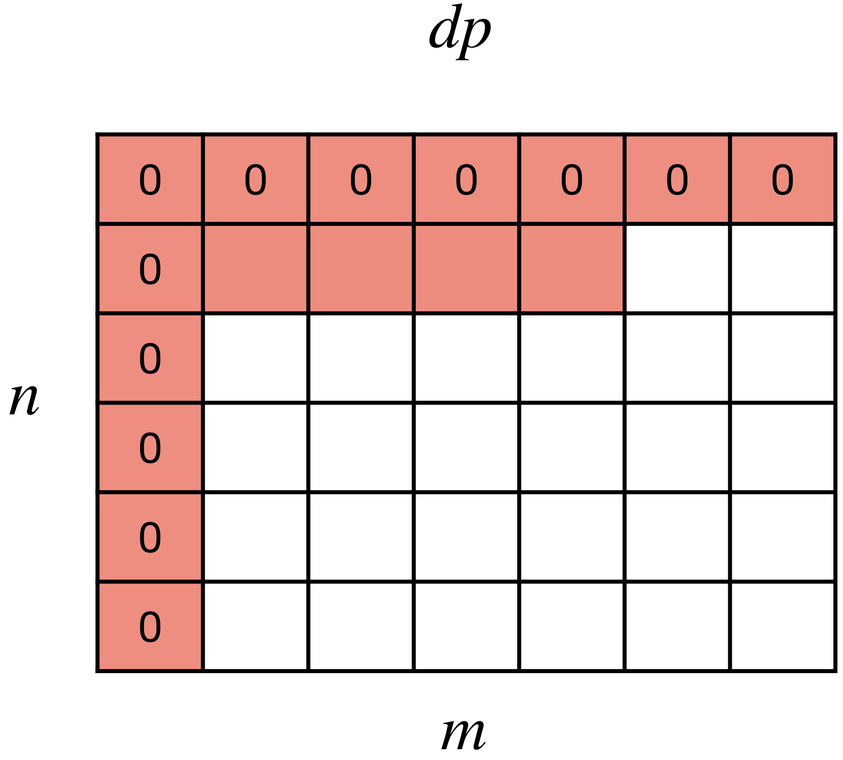

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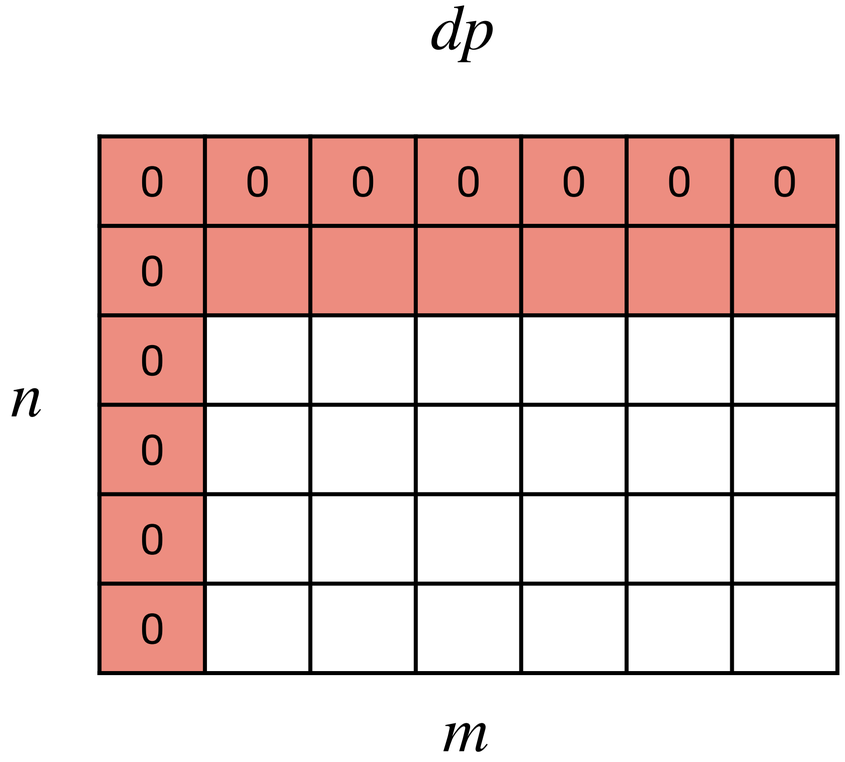

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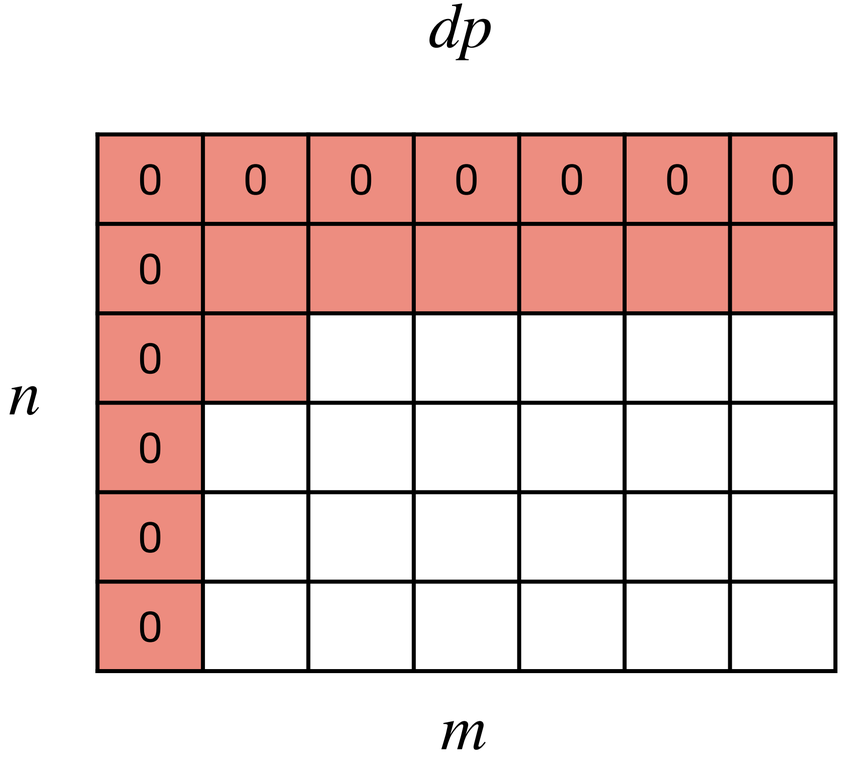

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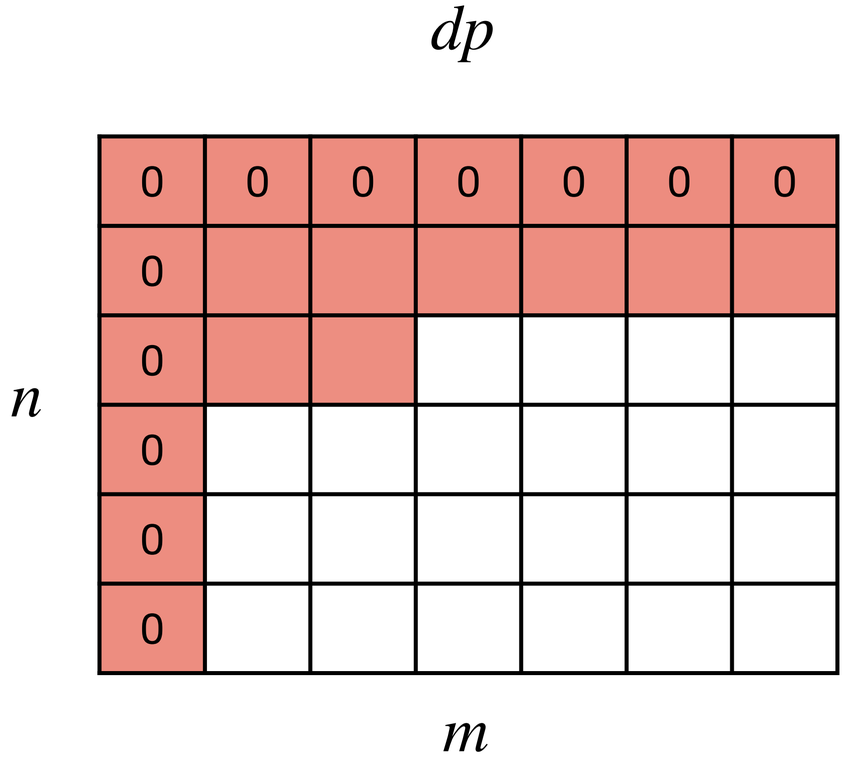

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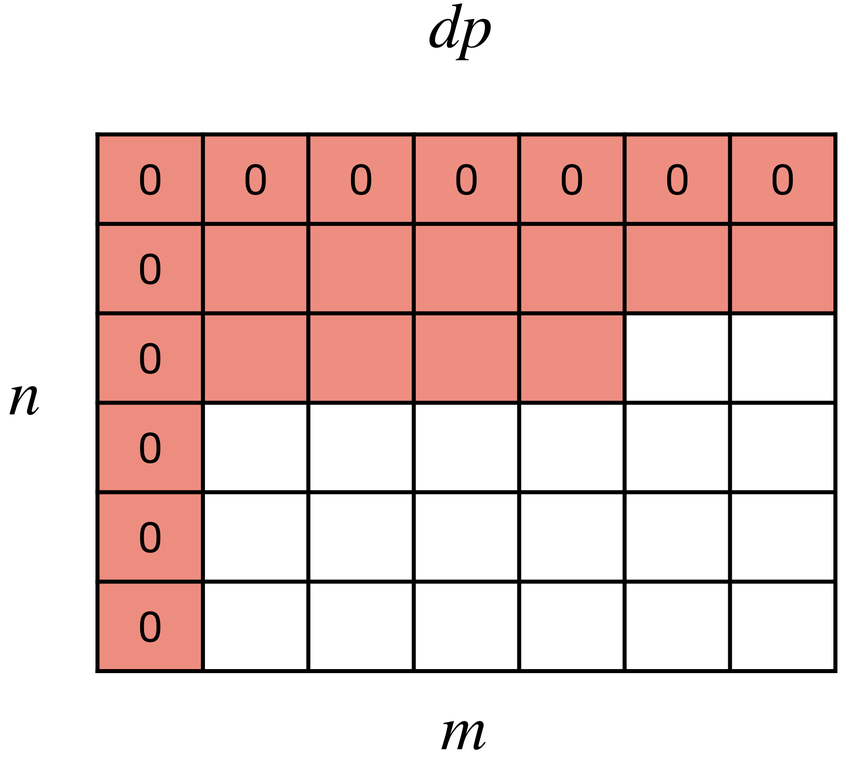

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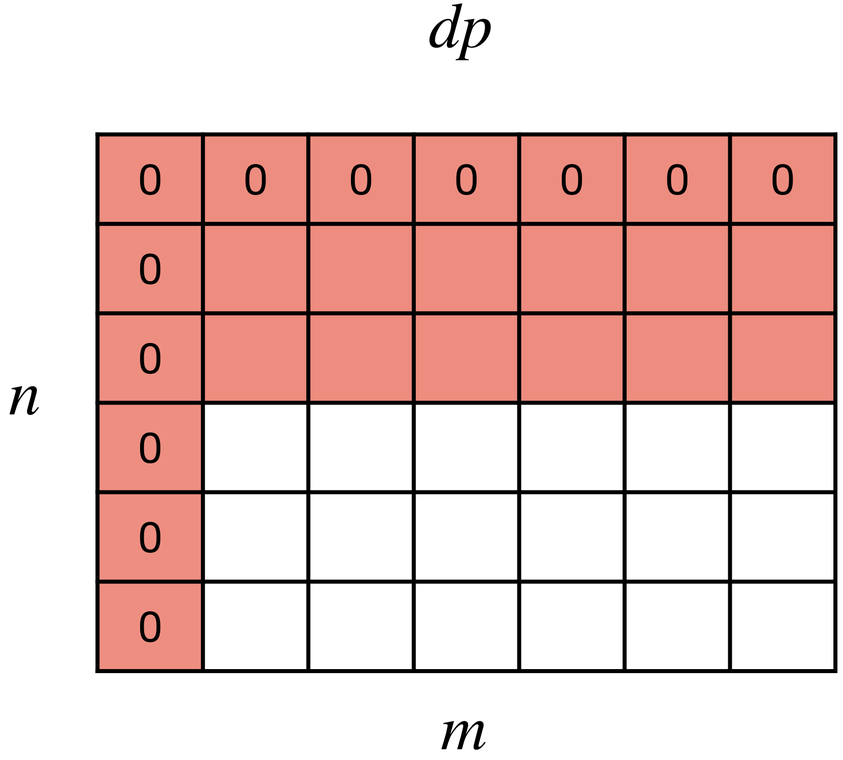

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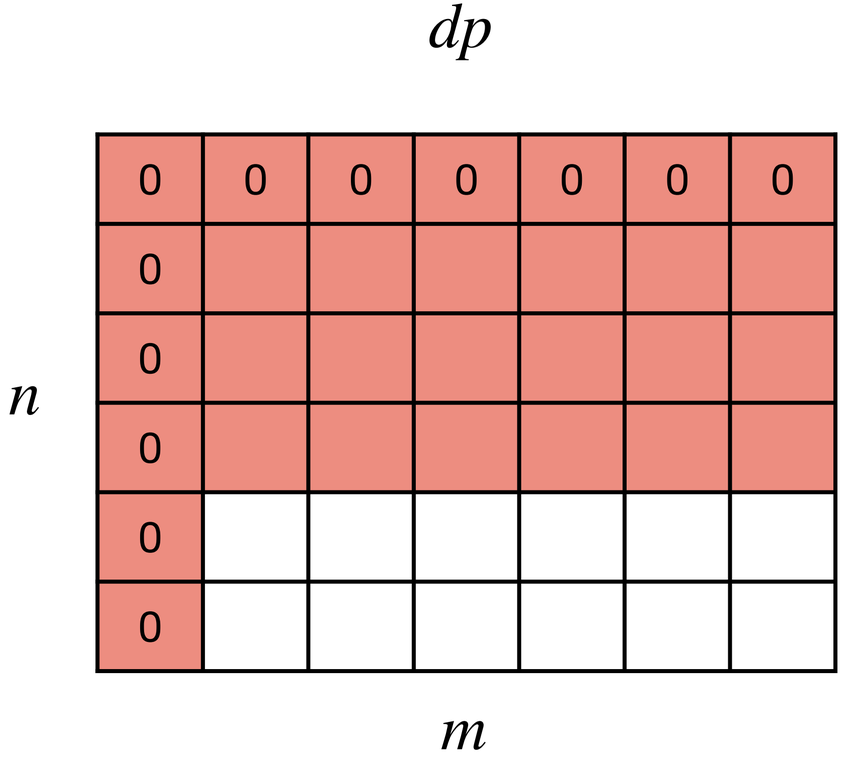

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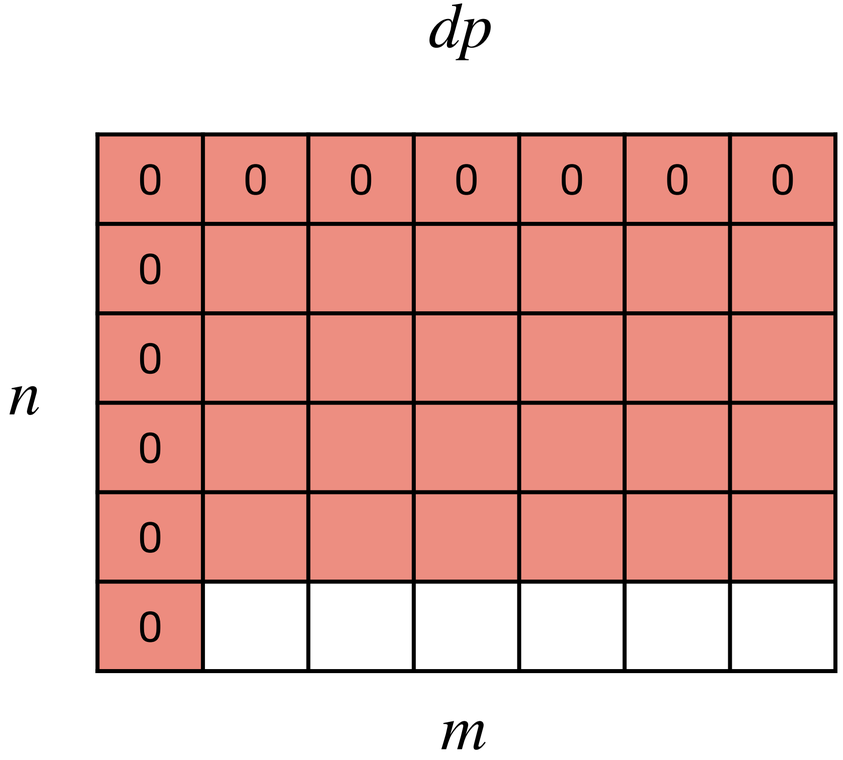

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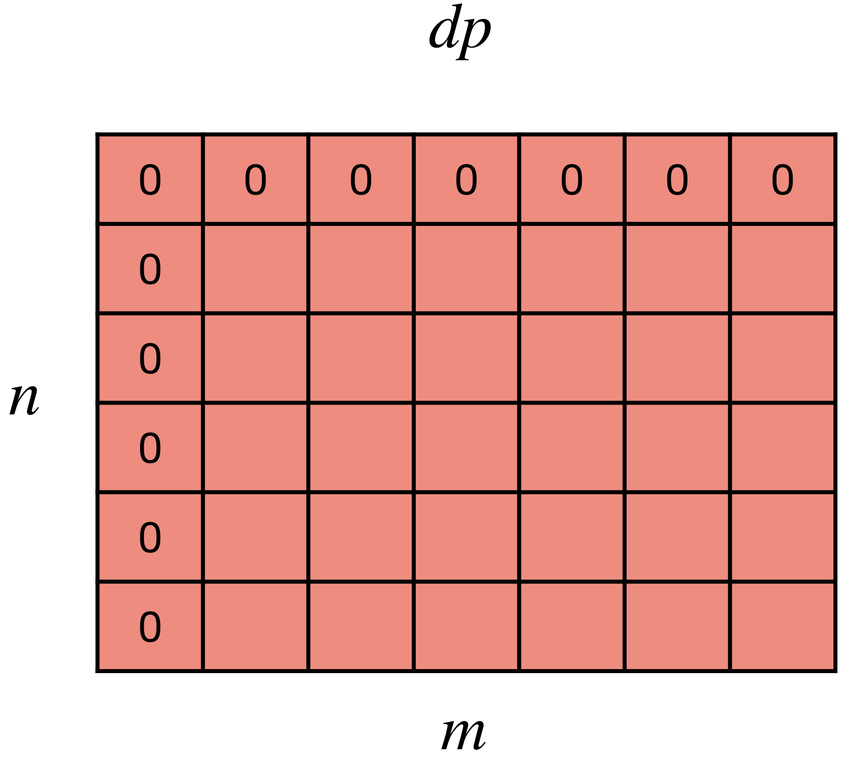

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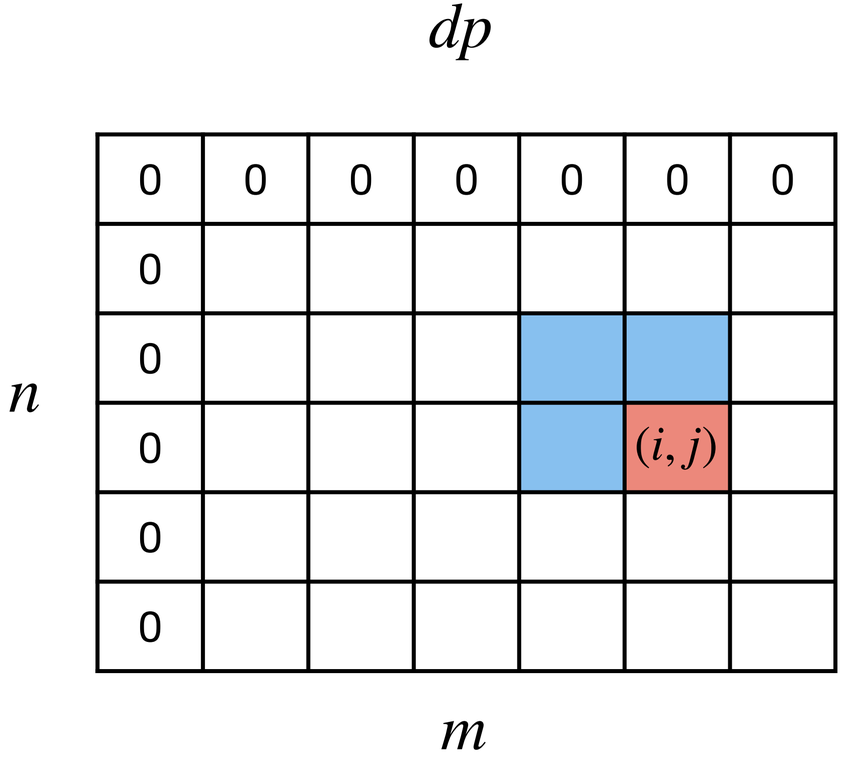

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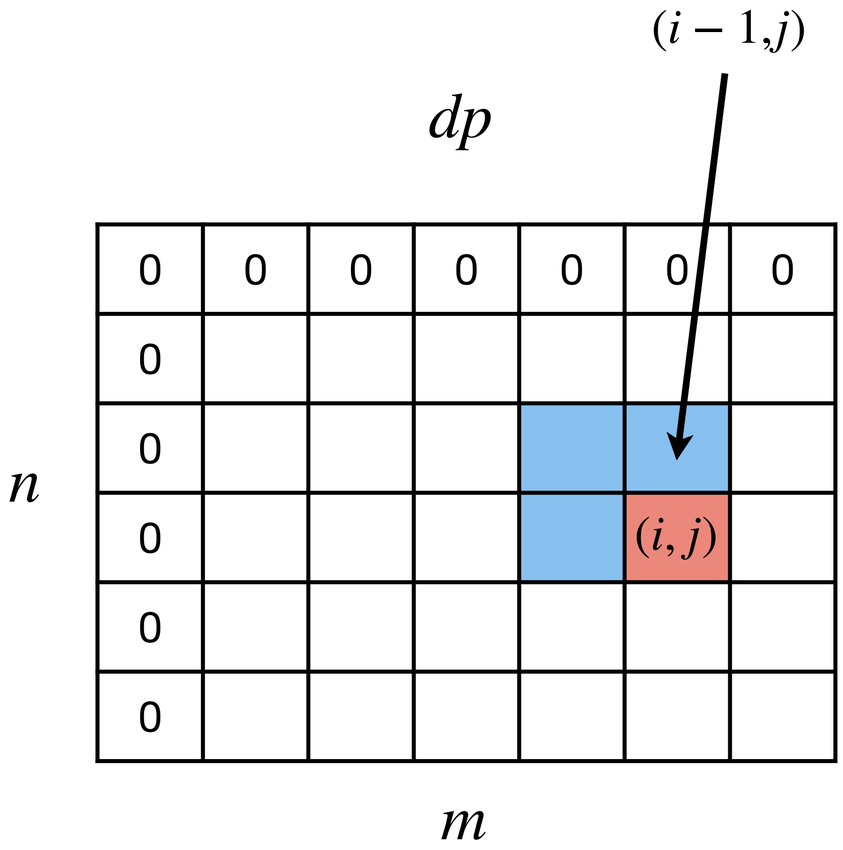

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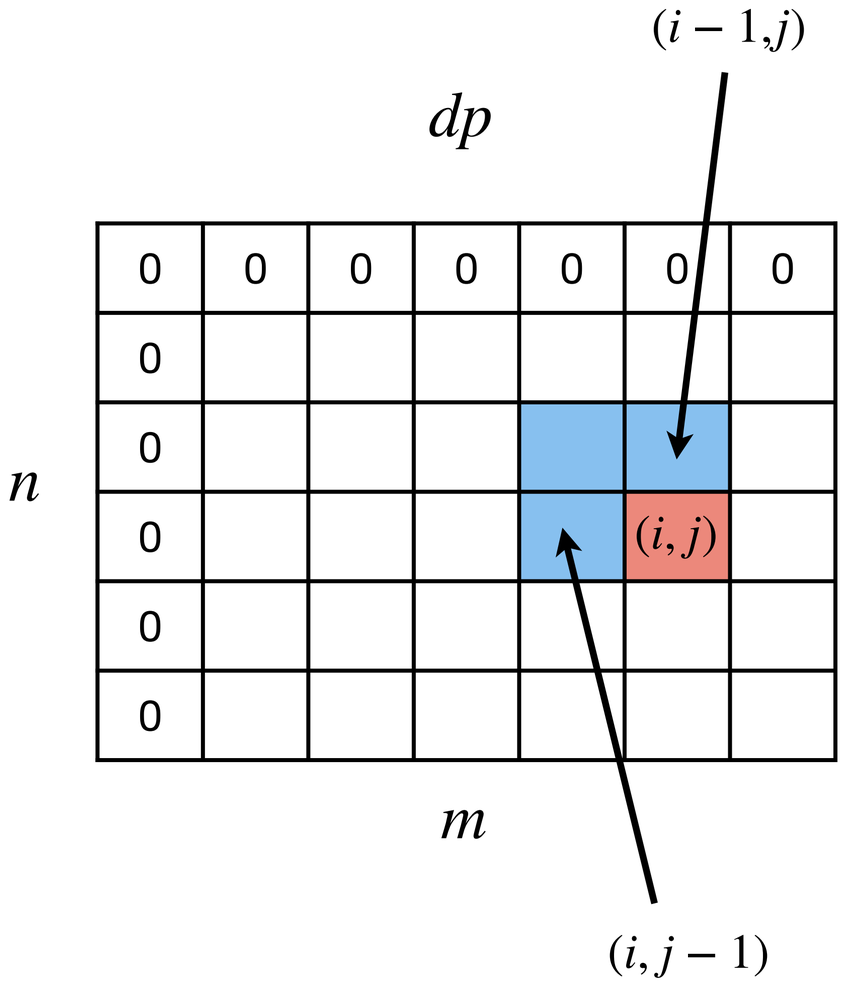

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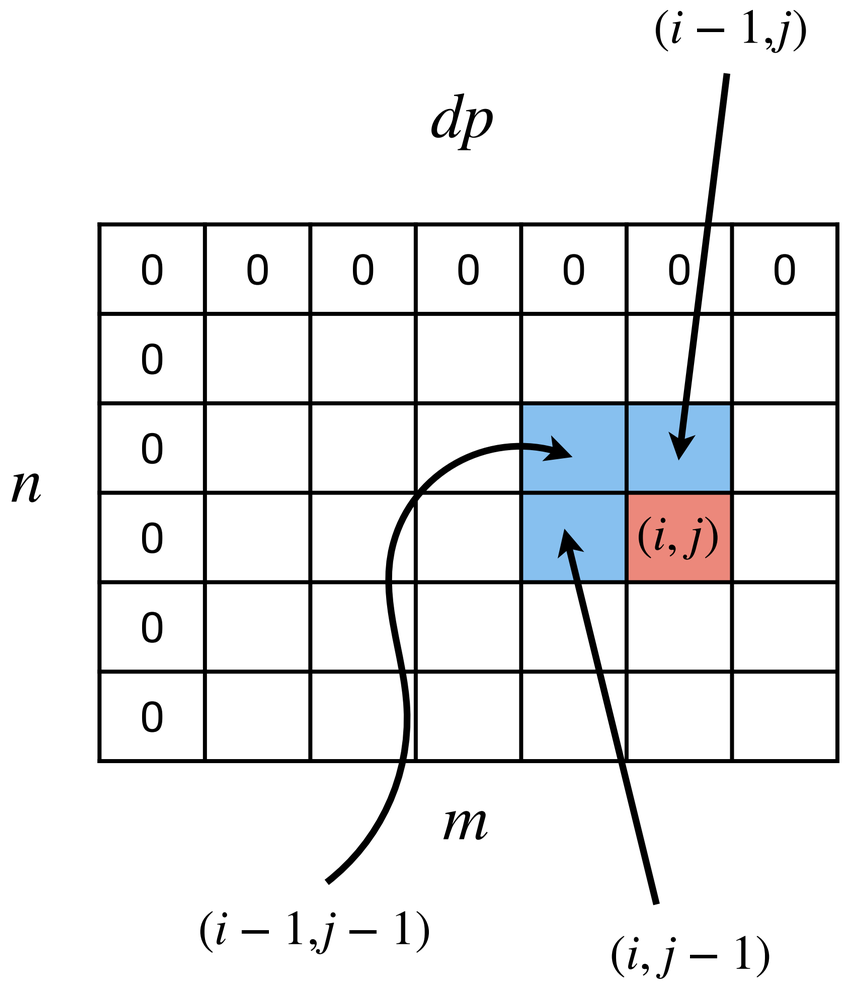

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Time Complexity: $\Theta(nm)$

Bottom-Up vs Top-Down DP

Bottom-Up vs Top-Down DP

Let $X =$ "aaa…a" and $Y =$ "aaa…a" be two sequences of n many as.

Bottom-Up vs Top-Down DP

Let $X =$ "aaa…a" and $Y =$ "aaa…a" be two sequences of n many as.

How will top-down LCS run on X and Y?

Bottom-Up vs Top-Down DP

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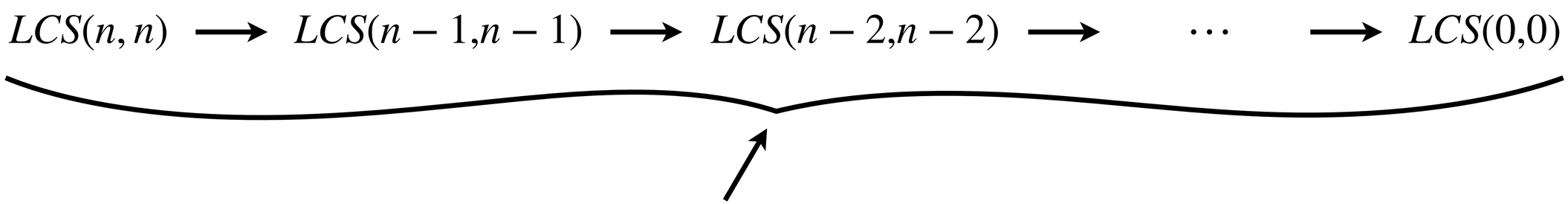

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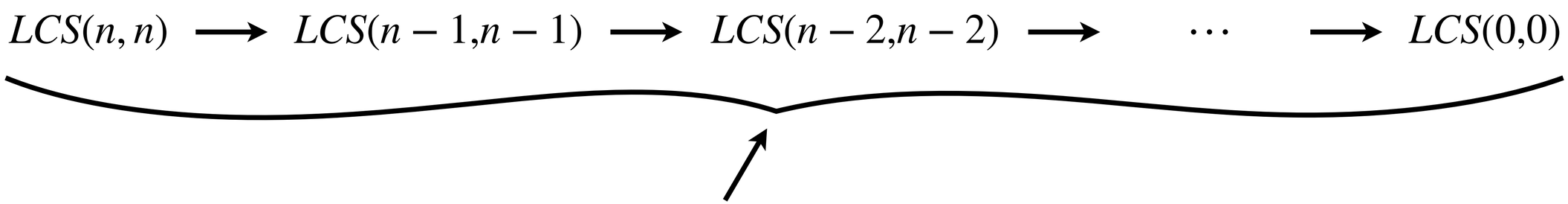

Top-down LCS requires n many recursive calls. Hence, runtime would be $\Theta(n)$.

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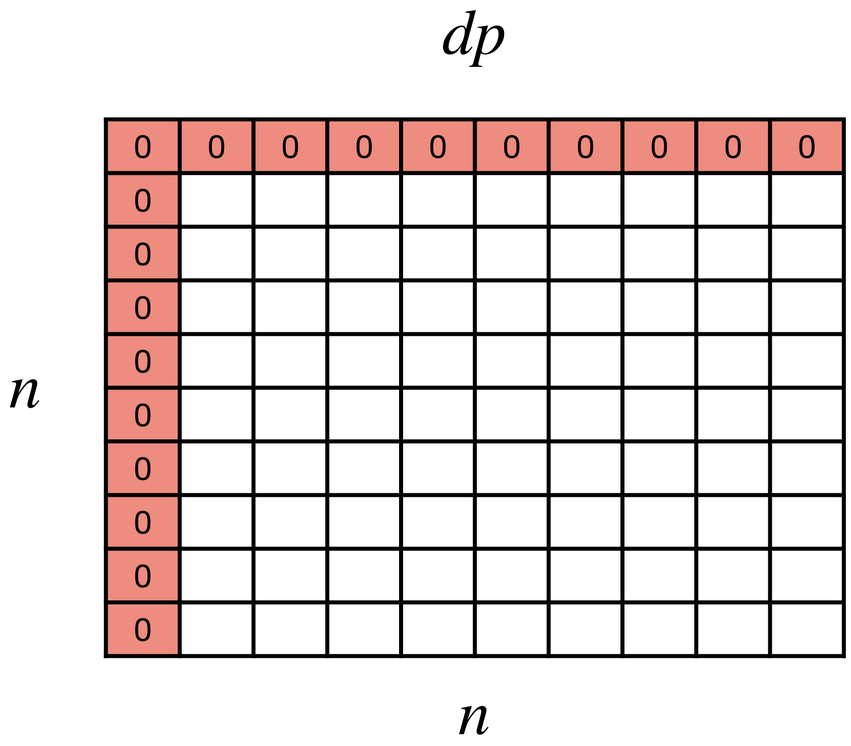

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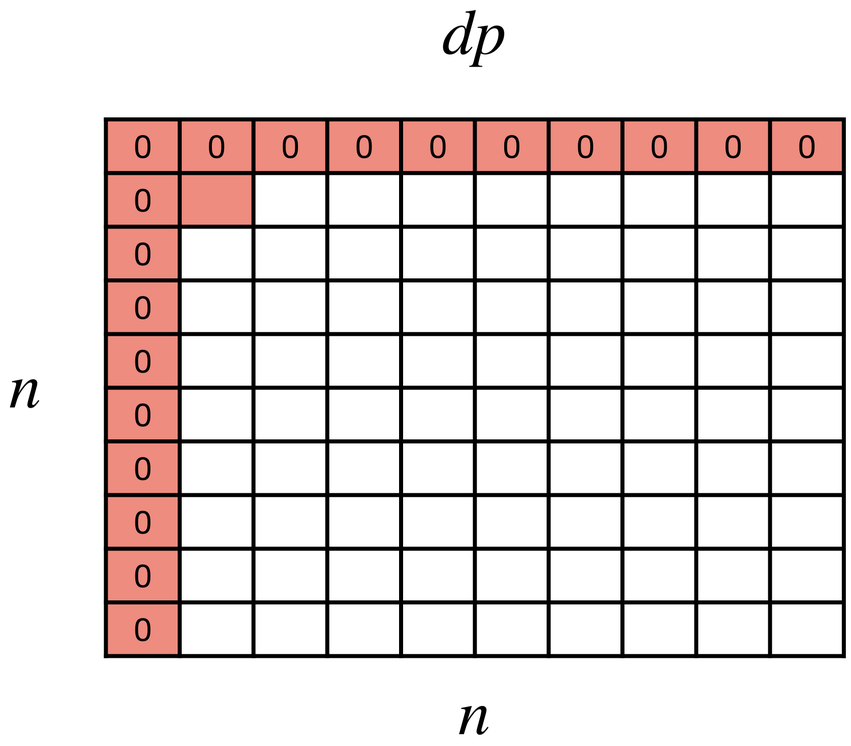

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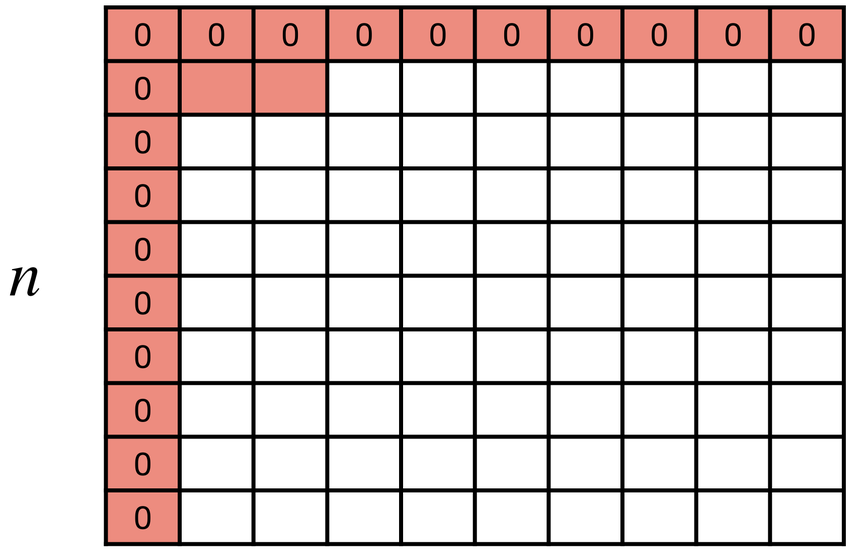
dp



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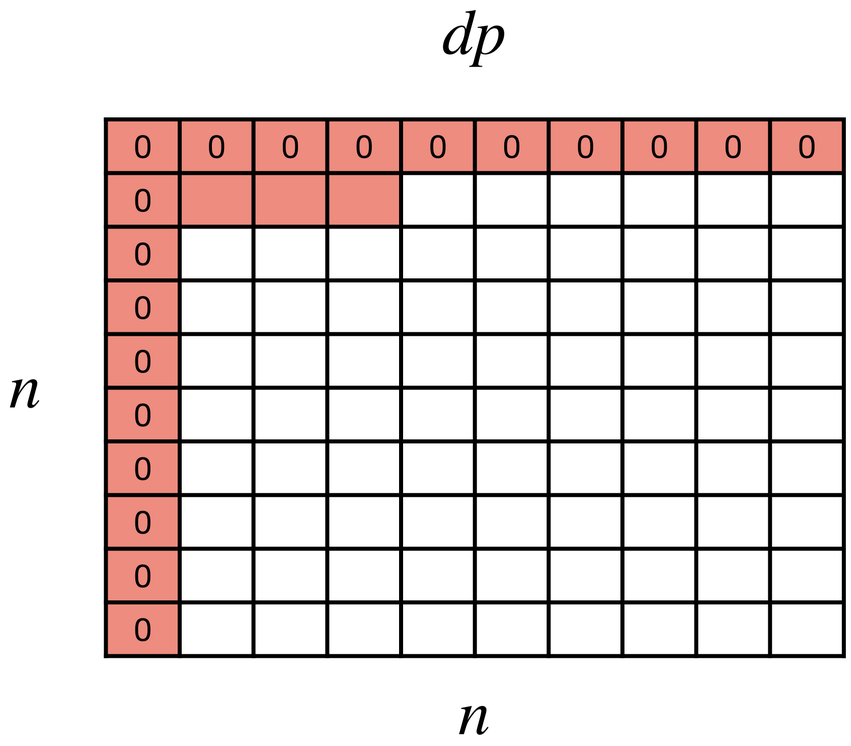

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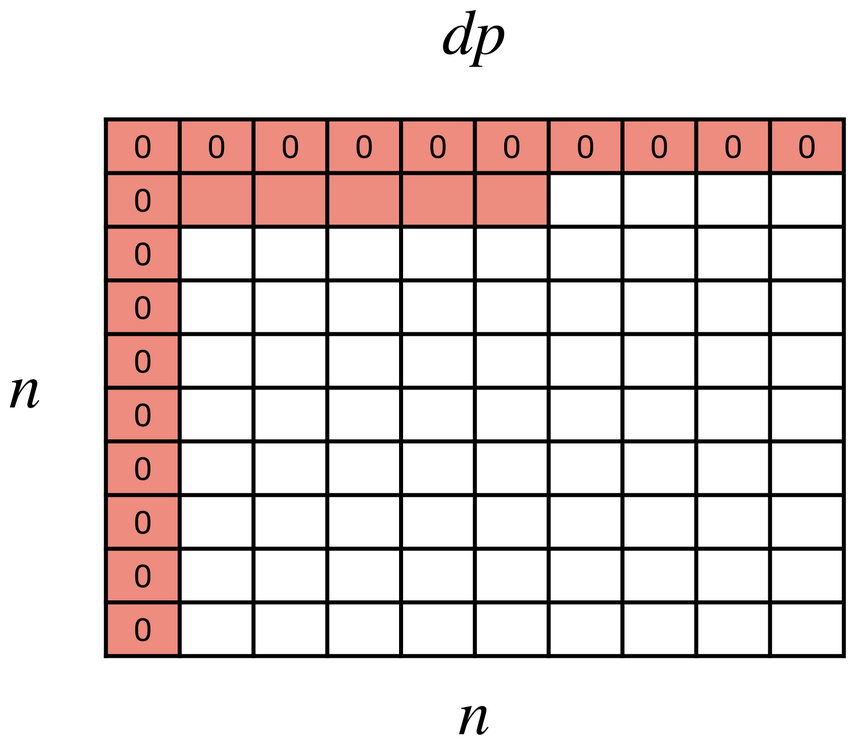

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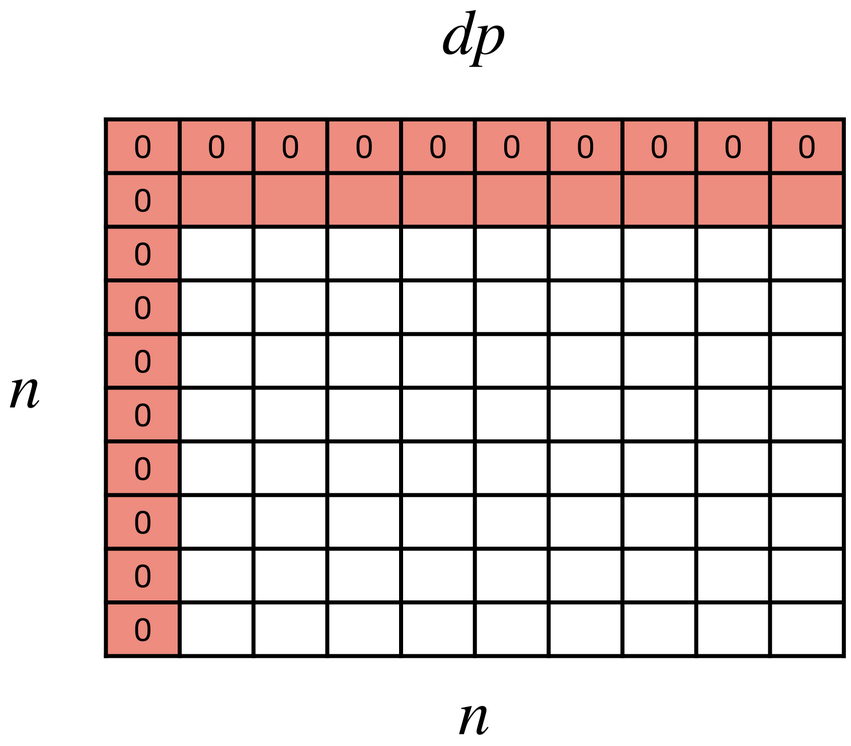

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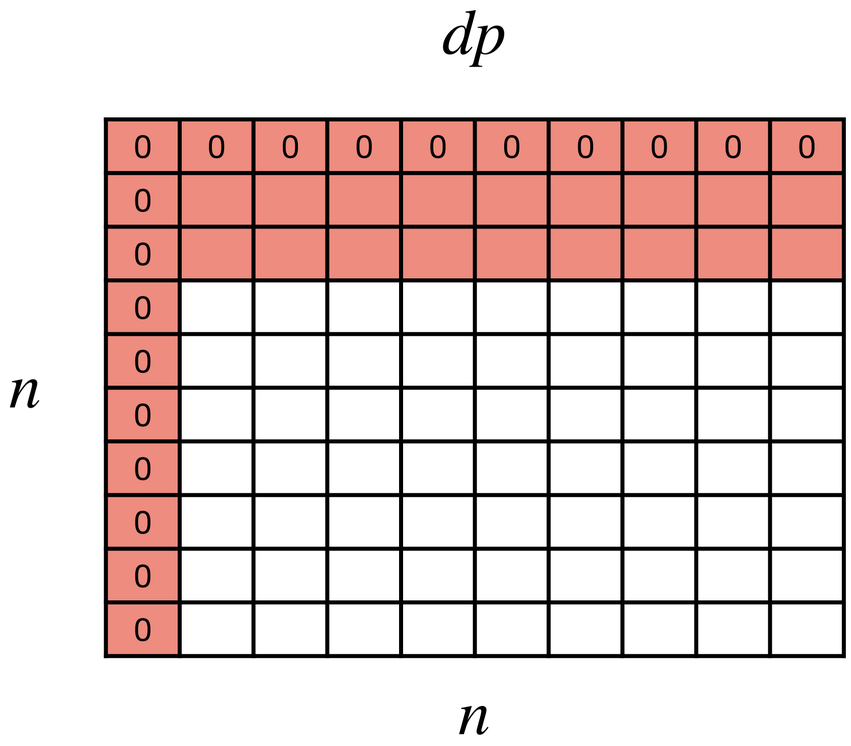

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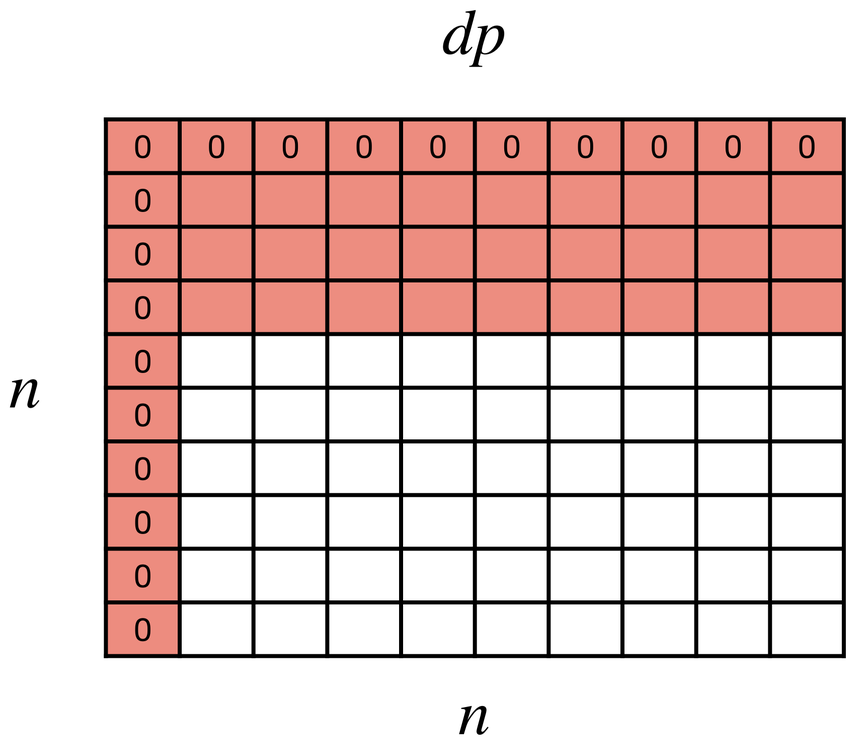

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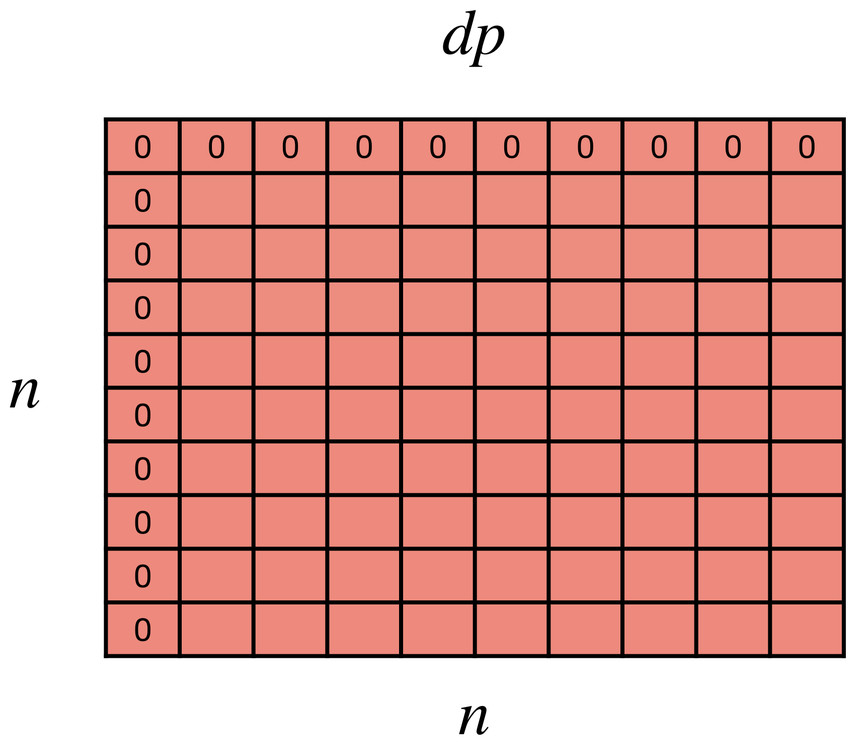

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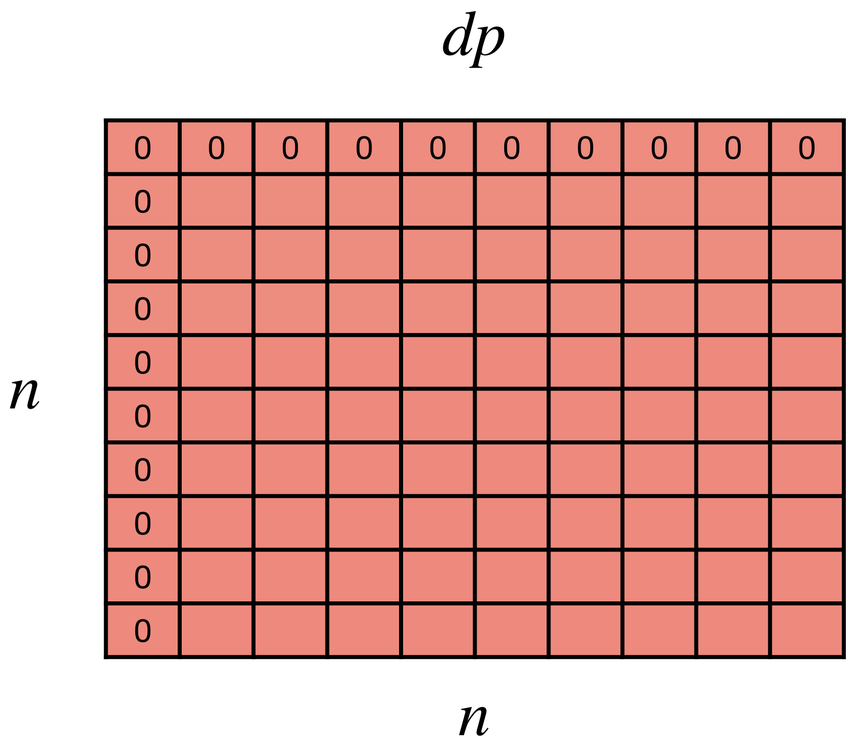

Bottom-up LCS will still fill all $n * n$ entries of dp array in $\Theta(n^2)$ time.

Bottom-Up vs Top-Down DP

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- If all subproblems need to be solved, then bottom-up is better due to no recursion overhead.
- Otherwise, top-down might be better.